AlgTop Final

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Solution 1

Let's begin by defining $G := \pi_1(X, x)$.

(1) By our assumptions on X (pc, lpc, slsc), we can invoke the Galois correspondence ensuring that it is sufficient to classify all conjugacy classes of subgroups in G (Thm 1.38). Thankfully, the only subgroups of G are the following:

$$\langle 0 \rangle = \{(0,0)\}\,, \quad \langle (1,0) \rangle = \{(0,0),(1,0)\} \cong \mathbb{Z}/2\,, \quad \langle (0,1) \rangle = \{(0,0),(0,1),(0,2)\} \cong \mathbb{Z}/3\,, \quad G$$

We will now take some time to prove this statement:

Proof. First, it hopefully goes without saying that the above four subsets are indeed subgroups. Then by Lagrange's Theorem, we know all nontrivial subgroups of G have order 2 or 3 (the only nontrivial divisors of $|G| = 6$). To finish the classification proof, we can check by hand that these are the only subgroups, or we can effortlessly invoke Sylow's second theorem: all order p subgroups of a given finite group are conjugate to each other when p is prime. Therefore, all other nontrivial subgroups (which necessarily have order 2 or 3) are conjugate to one of the two we have already found. However, G is abelian, so our subgroups are closed under conjugation and thus, the only order 2 (or 3) group is $\langle (1,0) \rangle$ (or $\langle (0,1) \rangle$).

□

In the above proof, we saw that the four subgroups are closed under conjugation, so there are four conjugacy classes of subgroups of G:

$$\{\langle 0 \rangle\}\,, \quad \{\langle (1,0) \rangle\}\,, \quad \{\langle (0,1) \rangle\}\,, \quad \{G\}$$

There are then four unique (up to iso) pc covering spaces of X which can all be obtained via the Galois correspondence. To explicitly capture these spaces, we could begin by constructing the universal cover as done in lecture and then quotienting to obtain the remaining covers. Of course, we cannot do this explicitly without more knowledge of X, but we can say for certain that the universal cover is the one corresponding to $\{\langle 0 \rangle\}$ and the cover corresponding to $\{G\}$ is the trivial one ($X \xrightarrow{\mathrm{id}_X} X$).

(2) Let $\mathscr{G}(H)$ be the covering space corresponding to the conjugacy class of the subgroup $H \subset G$ (via the Galois correspondence) and $\deg \mathscr{G}(H)$ its degree. We know from lecture that $\deg \mathscr{G}(H) = |G/H| = |G|/|H|$. This was because each fiber is iso to G/H as a right G-set. Then the computations are simple:

$$\deg \mathscr{G}(\langle 0 \rangle) = 6/1 = 6\,, \quad \deg \mathscr{G}(\langle (1,0) \rangle) = 6/2 = 3\,,$$
$$\deg \mathscr{G}(\langle (0,1) \rangle) = 6/3 = 2\,, \quad \deg \mathscr{G}(G) = 6/6 = 1$$

(3) Recall the following correspondence $\text{Deck}(\mathscr{G}(H)) \cong N_G(H)/H$ where

$$N_G(H) = \{g \in G \mid gHg^{-1} \subset H\}$$

is the normalizer of H in G. As explained in *Part (1)*, G being obviously abelian implies that all subgroups are closed under conjugation and hence, $N_G(H) = G$ for all subgroups H. Note that this is equivalent to stating that H is normal in G, so $N_G(H)/H = G/H$ is a well-defined group. We conclude with the following calculations:

$$\begin{gathered}\text{Deck}(\mathscr{G}(\langle 0\rangle)) \cong G/\langle 0\rangle = G \\ \text{Deck}(\mathscr{G}(\langle(1,0)\rangle)) \cong G/\langle(1,0)\rangle \cong \mathbb{Z}/3 \\ \text{Deck}(\mathscr{G}(\langle(0,1)\rangle)) \cong G/\langle(0,1)\rangle \cong \mathbb{Z}/2 \\ \text{Deck}(\mathscr{G}(G)) \cong G/G \cong \mathbb{Z}/1\end{gathered}$$

The middle two calculations are the only nontrivial ones. They both follow from either direct inspection, or the observation that $|G/\langle(1,0)\rangle| = 3$, $|G/\langle(0,1)\rangle| = 2$, and the fact that there is a unique (up to iso) group of order p when p is prime.

Solution 2

Given an n-manifold M and an element $m \in M$, we have a some neighborhood U of m on which we have a local chart $\chi_U : U \to \mathbb{R}^n$. In particular, χ_U is a homeomorphism onto its image (which is necessarily an open in $\mathbb{R}^n$), so $\chi_U(U - \{m\}) = \chi_U(U) - \chi_U(\{m\})$. Further, $\chi_U(U) - \chi_U(\{m\})$ is open in $\mathbb{R}^n$ implying $U - \{m\}$ is open in M, [1]. Then also $M - \{m\} = M \cap (U - \{m\})$ is open in M. Setting $A := M - \{m\}$ and $B := U$, we see that the interiors of A and B (which are simply A and B) cover M. Therefore, we can invoke the Excision Theorem (Thm 2.20) and conclude that

$$H_k(B, A \cap B) \cong H_k(M, A)$$

for all k. Replacing A and B with their definitions, we then see that the above is precisely stating

$$H_k(U, U - \{m\}) \cong H_k(M, M - \{m\})$$

for all k. Then because χ_U is a homeomorphism and homology is invariant under such maps, we also have the following:

$$H_k(U, U - \{m\}) \cong H_k(\chi_U(U), \chi_U(U) - \{\chi_U(m)\})$$

Because $\chi_U(U)$ is open in $\mathbb{R}^n$ and $\chi_U(m) \in \chi_U(U)$, we can invoke the result in the proof of Thm 2.26[2] stating that for any open $V \subset \mathbb{R}^n$ and $x \in V$, we have

$$H_k(V, V - \{x\}) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & o.w. \end{cases}$$

Thus, by transitivity of $\cong$, we have the same result for $H_k(M, M - \{m\})$.

□

We can then conclude that an n_1-manifold M_1 and an n_2-manifold M_2 cannot be homeomorphic if $n_1 \neq n_2$. This is because, again, homology is invariant under homeomorphism, so such a homeomorphism, denoted ψ, would imply that $H_k(M_1, M_1 - \{x\}) \cong H_k(M_2, M_2 - \{\psi(x)\})$ for all k, which is a clear contradiction of the previous result.

[1] Perhaps more accurately, $\chi_U(U) - \{\chi_U(m)\}$ is open in $\mathbb{R}^n$ implying that it is open in the subspace topology on $\chi_U(U)$. From this, we have that $U - \{m\}$ is open in the subspace topology on U, by χ_U a homeomorphism between these topologies. Then, $U - \{m\}$ will be open in M because the defining subspace, U, is open.

[2] For more details, see the remark below.

Remark. To prove the intermediate result that I invoke from the proof of Thm 2.26, Hatcher vaguely says that something is true by inspecting the long exact sequence on the pair $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$. The desired result is then immediately clear in all cases $k > 1$, but the last two cases require further considerations. I did work through the full proof on a blackboard and can send a photo if you would like.

Solution 3

First, let's get three necessary lemmas out of the way. Once this is done, we will show that $\mathbb{R}P^2 \vee S^2$ is path connected (pc), locally path connected (lpc), and semi-locally simply connected (slsc). To prove these lemmas it will be helpful to introduce some notation. Recall that $X \vee Y := X \amalg_\sim Y$ where $\sim$ identifies the wedge point $w_X \in X$ with that in Y; the resulting point will be denoted $w := w_X \sim w_Y \in X \vee Y$. Further, let $q : X \amalg Y \to X \vee Y$ denote the projection (sending $w_X \mapsto w \leftarrow w_Y$). Finally, denote the inclusion $\iota_Z : Z \to X \amalg Y$ for $Z = X, Y$.

Lemma 1 (Wedge pc)**.** If X and Y are pc, then $X \vee Y$ is pc.

Proof. Suppose X, Y path connected and take any $a, b \in X \vee Y$. If $a, b \in \text{im}(q\iota_X) \cong X$ or $a, b \in \text{im}(q\iota_Y) \cong Y$ then obviously we have a path from a to b pushed forward from a path in the respective space X or Y. For the other case, assume $a \in \text{im}(q\iota_X) \cong X$ and $b \in \text{im}(q\iota_Y) \cong Y$. Then we have paths γ_a and γ_b from a and b to the wedge point $w = \text{im}(q\iota_X) \cap \text{im}(q\iota_Y)$, respectively. Then $\gamma_a\overline{\gamma_b}$ is a path from a to b. □

Lemma 2 (Wedge lpc)**.** If X and Y are lpc, then $X \vee Y$ is lpc.

Proof. To see this, consider some point $z \in X \vee Y$ and neighborhood U of z. U can be rewritten as $U_X \amalg_\sim U_Y)$ and we will assume WLOG that $U_X \amalg U_Y$ is a "good representative" of $U_X \amalg_\sim U_Y$ meaning $U_X \amalg U_Y = q^{-1}(U_X \amalg_\sim U_Y)$. In particular, this means that if $w \in U_X \amalg_\sim U_Y$, then $w_X \in U_X$ and $w_Y \in U_Y$. We will first prove the result for the case when $z = w$.

(Case z=w) In this case, we have $w_X \in U_X$ by our "good representative" assumption which yields a pc open $V_X \subset U_X$ such that $w_X \in V_X^w$ and the analogous result for Y. Then $z = w \in V_X^w \amalg_\sim V_Y^w \subset U_X \amalg_\sim U_Y$. Further, by *Lemma 1*, we have that $V_X^w \amalg_\sim V_Y^w = V_X^w \vee V_Y^w$ is pc.

(General case) We can assume WLOG that $z \in X$. Then there is a pc open $V_X \subset U_X$ with $z \in V_X$. Now, we will again break up into two cases:

> *(Case $w_X \notin V_X$)* If $w_X \notin V_X$, then $q^{-1}(V_X \amalg_\sim \emptyset) = V_X \amalg \emptyset$ which is open in $X \amalg Y$ and thus, by definition of quotient topology, $V_X \amalg_\sim \emptyset$ open in $X \vee Y$. Further, $V_X \amalg_\sim \emptyset \cong V_X$ is obviously pc and we are done.
>
> *(Case $w_X \in V_X$)* If $w_X \in V_X$, the previous proof doesn't work because $q^{-1}(V_X \amalg_\sim \emptyset) = V_X \amalg \{w_Y\}$ which is not necessarily open in $X \amalg Y$. To get around this, we can consider $V_X \amalg_\sim V_Y^w$ where V_Y^w is again a pc neighborhood of w_Y contained in U_Y. Then we again invoke *Lemma 1* to ensure that $V_X \amalg_\sim V_Y^w = V_X \vee V_Y^w$ is pc.

□

Lemma 3 (Wedge slsc)**.** If X and Y are slsc, then $X \vee Y$ is slsc.

Proof. Omitted, but follows that same logic as that of the previous lemma. □

Lemma 4 (Epi pc)**.** If $s : X \to Y$ is a continuous surjection, then X pc implies Y pc.

Proof. We can take two points $y, y' \in Y$ and any x, x' in $s^{-1}(y), s^{-1}(y')$, respectfully. We then have a path γ from x' to y', by X path connected. Further, the composition $s\gamma$ will be a continuous path from $s\gamma(0) = sx = y$ to $s\gamma(1) = sx' = y'$. □

Lemma 5 (Epi lpc)**.** If Y is endowed with the quotient topology coming from a surjection $s : X \to Y$, then X lpc implies Y lpc.

Proof. Consider a neighborhood U of $y \in Y$ and its pullback $s^{-1}(U)$ which is necessarily open in X. By X lpc, we have we have a pc neighborhood V of x for some choice $x \in s^{-1}(y)$ satisfying $V \subset s^{-1}(U)$. Then $s(V) \subset U$ is necessarily open in Y (by definition of quotient topology) and $y \in s(V)$. Further, by *Lemma 4*, $s(V)$ is pc. □

Lemma 6 (Epi slsc)**.** If Y is endowed with the quotient topology coming from a surjection $s : X \to Y$, then X slsc implies Y pc.

Proof. Omitted, but follows the same logic as that of the previous lemma. □

With these lemmas out of the way, we can now easily show that $\mathbb{R}P^2 \vee S^2$ is path connected, locally path connected, and semi-locally simply connected. For this, we will frequently use that $\mathbb{R}P^2$ is endowed with the quotient topology from the defining projection $\pi : S^2 \to \mathbb{R}P^2$ (by identifying antipodal points). We know that S^2 is pc, lpc, & slsc, so by *Lemmas 4, 5, & 6* we have that $\mathbb{R}P^2$ is pc, lpc, & slsc. Then we can invoke *Lemmas 1, 2, & 3* and we have the desired result that the $S^2 \vee \mathbb{R}P^2$ is pc, lpc, & slsc.

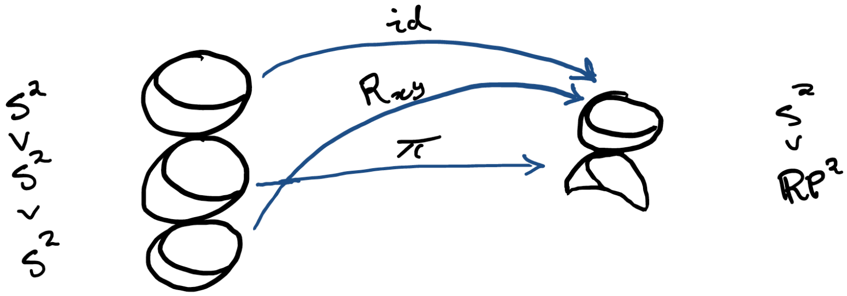
With this result, we know that we can readily employ the Galois correspondence (Thm 1.38) to assist us in finding all pc covers of $S^2 \vee \mathbb{R}P^2$. By the Corollary to Van Kampen's Theorem discussed in class, we have $\pi_1(\vee_i X_i, x) \cong *_i \pi_1(X_i, x)$. Using this in our case, we get

$$\pi_1(\mathbb{R}P^2 \vee S^2, x) \cong \pi_1(\mathbb{R}P^2, x) * \pi_1(S^2, x) \cong \mathbb{Z}/2 * \{e\} \cong \mathbb{Z}/2$$

The last equivalence holds because the free product between any group G and the trivial group is obviously just G. Therefore, by the Galois correspondence, we have that the only unique (up to iso) covering spaces of $\mathbb{R}P^2 \vee S^2$ are the two conjugacy classes of subgroups of $\mathbb{Z}/2$, namely $\{0\}$, and $\mathbb{Z}/2$ itself. The $\{0\}$ conjugacy class will always correspond to the universal cover, and the whole group $\mathbb{Z}/2$ always corresponds to the trivial cover, in this case $\mathbb{R}P^2 \vee S^2 \xrightarrow{\text{id}} \mathbb{R}P^2 \vee S^2$. It remains to find the universal cover. Because $|(\mathbb{Z}/2)/\{0\}| = 2$, it is sufficient to find some degree two cover and this will necessarily be the universal cover. Consider the degree two cover below (image will follow):

$$S^2 \vee S^2 \vee S^2 \xrightarrow{p} \mathbb{R}P^2 \vee S^2$$

To define p, suppose that the wedge $S^2 \vee S^2 \vee S^2$ is stacked vertically meaning the north pole of the left sphere is wedged to the south pole of the middle and the north pole of the middle sphere is wedged to the south pole of the right sphere. We further assume without any loss of generality that $\mathbb{R}P^2 \vee S^2$ is wedged similarly where the pseudo-north pole of $\mathbb{R}P^2$ (the image of the north pole under π) is wedged to the south pole of S^2. Then $p = p_l \vee p_m \vee p_r$ where p_m sends the middle copy of S^2 in the domain to the $\mathbb{R}P^2$ in the codomain via π. This ensures that the two wedge points in the domain get mapped to the one wedge point in the codomain. Then p_r can act as identity between the right copy of S^2 in the domain and the copy of S^2 in the codomain. Finally, p_l maps the left copy of S^2 in the domain to the copy of S^2 in the codomain, but it must agree with p_m at the left wedge point, so p_l flips S^2 upside down sending the north pole to the south pole and vise-versa (we call this map R_{xy} below). Pictorially, we have the following:



This construction is pretty clearly a covering space and obviously of degree two. The only point we need to check is at the wedge in the codomain. If we take an open neighborhood of this point which is small enough, it will just look like two surfaces wedged at a point and it will indeed be homeomorphic to the two disjoint opens in its preimage. We can then conclude that this is indeed the universal cover.

Solution 4

Throughout this problem, let us denote e_0 as the zero vertex in Δ^n. Also before we begin, I will present a theorem which is a combination of a theorem and its corollary presented in lecture:

Theorem 1. *Consider the following data where p is some covering map (not necessarily universal).*

$$\begin{array}{ccc} & & (\widetilde{X}, \widetilde{x}) \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow p \\ (Y, y) & \xrightarrow[f]{} & (X, x) \end{array}$$

If Y is path connected and locally path connected, then there is such a lift $\widetilde{f}$ if and only if $f_*(\pi_1(Y, y)) \subset p_*(\pi_1(\widetilde{X}, \widetilde{x}))$.

All parts of this problem will follow relatively quickly from this theorem. When deferring to this theorem, we will not explicitly check nor even mention that the relevant spaces are path connected and locally path connected because all spaces discussed in this problem are.

(1) Consider some $\sigma \in \mathrm{Map}(\Delta^n, X)$ and let $\widetilde{x} \in p^{-1}(\sigma(e_0))$. Because $\widetilde{X}$ is the universal cover, $p_*(\pi_1(\widetilde{X}, \widetilde{x})) = p_*(c_{\widetilde{x}}) = c_{p(\widetilde{x})} = c_{\sigma(e_0)}$. Further, Δ^n has trivial fundamental group because it is contractible and thus, $\sigma_*(\pi_1(\Delta^n, e_0)) = c_{\sigma(e_0)}$. Therefore, by *Theorem 1*, there exists $\widetilde{\sigma} : (\Delta^n, e_0) \to (\widetilde{X}, \widetilde{x})$ such that $p\widetilde{\sigma} = \sigma$. In fact, there are as many lifts of σ as there are choices for $\widetilde{x}$. In our case of the universal cover, there are $|G|$ choices for $\widetilde{x}$, so $p_* : \mathrm{Map}(\Delta^n, \widetilde{X}) \to \mathrm{Map}(\Delta^n, X)$ is a degree $|G|$ cover of its codomain (at least as a set-morphism).

□

(2) For this problem, we will use f_g to denote the deck transformation associated to $g \in G$ as we did in lecture. Because $\widetilde{X}$ is the universal cover, $\mathrm{Deck}(\widetilde{X}) \cong N_G(e)/e = G/e \cong G$, so there is an f_g for all g.

Lemma 7. $f_e = \mathrm{id}_{\widetilde{X}}$

Proof. This is quite clear from the definition. f_e is constructed, using *Theorem 1*, as the unique map $f : (\widetilde{X}, \widetilde{x}) \to (\widetilde{X}, e\widetilde{x})$ such that $pf = p$. Here $\widetilde{x} \in p^{-1}(x)$ is chosen arbitrarily. Because $e\widetilde{x} = \widetilde{x}$ we have that $\mathrm{id}_{\widetilde{X}}$ satisfies this requirement and by uniqueness, we are done. □

Lemma 8. If $f_g(y) = y$ for some $y \in \widetilde{X}$, then $g = e$.

Proof. Suppose $f_g(y) = y$. Let's begin with defining the pointed map $p : (\widetilde{X}, y) \to (X, p(y))$. Because the fundamental group of $\widetilde{X}$ is trivial, we have the condition necessary to invoke *Theorem 1* stating that there is a unique map $f : (\widetilde{X}, y) \to (\widetilde{X}, y)$ such that $pf = p$. We know that $\mathrm{id}_{\widetilde{X}}$ is one such lift, so by uniqueness,

$$f_g = f = \mathrm{id}_{\widetilde{X}} = f_e$$

where the last equality holds by the previous lemma. □

We can now swiftly prove the desired result:

($\Longleftarrow$) Suppose $g = e$. Then of course $f_g\widetilde{\sigma} = \widetilde{\sigma}$ because, by *Lemma 7*, $f_g = \mathrm{id}_{\widetilde{X}}$.

□

($\Longrightarrow$) Suppose $f_g\widetilde{\sigma} = \widetilde{\sigma}$. Then $f_g(\widetilde{\sigma}(e_0)) = \widetilde{\sigma}(e_0)$. Setting $y := \widetilde{\sigma}(e_0)$ we can defer to *Lemma 8* to conclude $f_g = f_e$.

□

(3) We know from lecture that the map $f_{(-)} : g \mapsto f_g$ gives us an isomorphism $G \cong \mathrm{Deck}(\widetilde{X})$ when $\widetilde{X}$ is the universal cover[3]. Just as well, we could have constructed $f_{(-)}$ by choosing different base points, say $\widetilde{\sigma}_1(e_0) \in \widetilde{X}$ and $p\widetilde{\sigma}_1(e_0) \in X$. Then we get the map

$$f^1_{(-)} : \pi_1(X, p\widetilde{\sigma}_1(e_0)) \to \mathrm{Deck}(\widetilde{X})$$

by sending γ to the unique $f : (\widetilde{X}, \widetilde{\sigma}_1(e_0)) \to (\widetilde{X}, \gamma\widetilde{\sigma}_1(e_0))$ such that $pf = p$. Note that $\widetilde{\sigma}_1(e_0), \widetilde{\sigma}_2(e_0) \in p^{-1}(\sigma(e_0))$ and because the fiber $p^{-1}(\sigma(e_0))$ is a transitive $\pi_1(X, p\widetilde{\sigma}_1(e_0))$-set, there is some γ such that $\gamma\widetilde{\sigma}_1(e_0) = \widetilde{\sigma}_2(e_0)$. Therefore, $f^1_\gamma \in \mathrm{Deck}(\widetilde{X})$ is a map

$$(\widetilde{X}, \widetilde{\sigma}_1(e_0)) \longrightarrow (\widetilde{X}, \widetilde{\sigma}_2(e_0))$$

It remains to show that $f^1_\gamma\widetilde{\sigma}_1 = \widetilde{\sigma}_2$. For this, we will use the fact that f^1_γ is a covering map of $\widetilde{X}$ which follows from it being a homeomorphism. Further, $\widetilde{\sigma}_{2_*}(\pi_1(\Delta^n), e_0) \subset f^1_{\gamma*}(\pi_1(\widetilde{X}, \widetilde{\sigma}_1(e_0)))$ because both fundamental groups are trivial and

$$f^1_{\gamma*}(c_{\widetilde{\sigma}_1(e_0)}) = c_{f^1_\gamma\widetilde{\sigma}_1(e_0)} = c_{\widetilde{\sigma}_2(e_0)} = \widetilde{\sigma}_{2*}(c_{e_0})$$

Then by *Theorem 1*, we have a unique δ fitting into the triangle below.

$$\begin{array}{ccc} & & (\widetilde{X}, \widetilde{\sigma}_1(e_0)) \\ & \overset{\delta}{\nearrow} & \downarrow f^1_\gamma \\ (\Delta^n, e_0) & \xrightarrow[\widetilde{\sigma}_2]{} & (\widetilde{X}, \widetilde{\sigma}_2(e_0)) \end{array}$$

We will now finish the proof by showing $\delta = \widetilde{\sigma}_1$ which will follow from δ being a lift of σ sending $e_0 \mapsto \widetilde{\sigma}_1(e_0)$ and the uniqueness of such lifts given by *Theorem 1*. We know that $\delta : e_0 \mapsto \widetilde{\sigma}_1(e_0)$, so to show that it is a lift of $\widetilde{\sigma}_1$, recall that f^1_γ being a deck transformation ensures that $pf^1_\gamma = p$ and observe the following calculation:

$$p\delta = pf^1_\gamma\delta = p\widetilde{\sigma}_2 = \sigma$$

Thus, $\delta = \widetilde{\sigma}_1$ and we are done by commutativity of the previous diagram.

□

[3]because the subgroup corresponding $\widetilde{X}$ is $\{e\}$ and then $N_G(e) = G$ and $N_G(e)/e \cong G$

Solution 5

It is necessary in this problem to assume $A \cap B$ is nonempty, so we will make this assumption.

By Theorem 2.13 and the explanation below it that (X, A) is a "good pair" when A is a subcomplex, we can consider the long exact sequence

$$\cdots \longrightarrow \widetilde{H}_k(A) \longrightarrow \widetilde{H}_k(X) \longrightarrow \widetilde{H}_k(X/A) \longrightarrow \widetilde{H}_{k-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{H}_0(X/A) \longrightarrow 0$$

By A contractible, we have that $\widetilde{H}_k(A) = 0$ for all k. This gives a collection of exact subsequences

$$0 \longrightarrow \widetilde{H}_k(X) \longrightarrow \widetilde{H}_k(X/A) \longrightarrow 0$$

implying that $\widetilde{H}_k(X) \cong \widetilde{H}_k(X/A)$. Then Proposition 2.22 gives us an iso $\widetilde{H}_k(X/A) \cong H_k(X, A)$ for all k and further, Corollary 2.24 yields $H_k(X, A) \cong H_k(B, A \cap B)$. Thus, $\widetilde{H}_k(X) \cong H_k(B, A \cap B)$. Investigating this latter group, we can see that $(B, A \cap B)$ is again a "good pair" because we $A \cap B$ is assumed to be nonempty, closedness of $A \cap B$ in B is inherited from that of A in X and the deformation retract of $N_\varepsilon(A)$ onto A (described around Proposition A.5 of the appendix) restricts to a deformation retract onto $A \cap B$ in B. Therefore, imposing Proposition 2.22 once again gives

$$\widetilde{H}_k(X) \cong H_k(B, A \cap B) \cong \widetilde{H}_k(B/A \cap B)$$

We can then consider the long exact sequence

$$\cdots \longrightarrow \widetilde{H}_k(A \cap B) \longrightarrow \widetilde{H}_k(B) \longrightarrow \widetilde{H}_k(B/A \cap B) \longrightarrow \widetilde{H}_{k-1}(A \cap B) \longrightarrow \cdots \longrightarrow \widetilde{H}_0(B/A \cap B) \longrightarrow 0$$

Just as before, we have that B is contractible, so $\widetilde{H}_k(B) = 0$ for all k implying $\widetilde{H}_k(B/A \cap B) \cong \widetilde{H}_{k-1}(A \cap B)$ for all k. Investigating the head of the long exact sequence above ($0 \to \widetilde{H}_0(B/A \cap B) \to 0$), we see that this equality still holds in the case $k = 0$ by defining $\widetilde{H}_{-1}(A \cap B) := 0$. Putting this together, we have

$$\widetilde{H}_k(X) \cong \widetilde{H}_k(B/A \cap B) \cong \widetilde{H}_{k-1}(A \cap B)$$

for all k. Then for all $k > 0$, we get the result

$$H_k(X) \cong \widetilde{H}_k(X) \cong \widetilde{H}_{k-1}(A \cap B)$$

There is still some subtlety at $k = 0$, because $H_0(X) \neq \widetilde{H}_0(X)$. In this case, the equation won't hold given our convention that $\widetilde{H}_{-1}(A \cap B) := 0$, but remember we set this convention so the reduced equation $\widetilde{H}_0(B/A \cap B) \cong \widetilde{H}_{-1}(A \cap B)$ holds. Therefore, $H_0(X) = \mathbb{Z}$ which can also be seen from the beginning by the fact that $X = A \cup B$ with A, B contractible and $A \cap B$ nonempty.

□