Category Theory Solution Set 5

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Solution 1

Suppose we have an adjunction, $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow FG$ and counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$. Then we can, for any category $\mathcal{J}$, define the functors $F_* : \mathcal{C}^{\mathcal{J}} \rightleftarrows \mathcal{D}^{\mathcal{J}} : G_*$ where $F_*K := F \circ K$ for all functors $K : \mathcal{J} \to \mathcal{C}$ (objects in $\mathcal{C}^{\mathcal{J}}$) and $F_*(\eta_j)_{j\in\mathcal{J}} := (F\eta_j)_{j\in\mathcal{J}}$ for all natural transformations $\eta = (\eta_j)_{j\in\mathcal{J}} : K \Rightarrow K'$ (morphisms in $\mathcal{C}^{\mathcal{J}}$). We use the analogous definition for G_*. Note that at times we will use FK for F_*K to make it clear that we are talking about composition of functors.

Remark 1. *F_* (and thus, G_*) is indeed a functor $\mathcal{C}^{\mathcal{J}} \to \mathcal{D}^{\mathcal{J}}$ (or $\mathcal{D}^{\mathcal{J}} \to \mathcal{C}^{\mathcal{J}}$ for G_*).*

Proof. The composition of two functors is again a functor, so $F_*(K)$ is defined in $\mathcal{D}^{\mathcal{J}}$ for all $K \in \mathcal{C}^{\mathcal{J}}$. Further, for any natural transformation $\eta : K \Rightarrow K'$ and $f : i \to j$ in $\mathcal{J}$, the diagram

$$\begin{array}{ccc} FKi & \xrightarrow{F\eta_i} & FK'i \\ {\scriptstyle FKf}\downarrow & & \downarrow{\scriptstyle FK'f} \\ FKj & \xrightarrow[F\eta_j]{} & FK'j \end{array} ,$$

is simply the image of

$$\begin{array}{ccc} Ki & \xrightarrow{\eta_i} & K'i \\ {\scriptstyle Kf}\downarrow & \circlearrowleft & \downarrow{\scriptstyle K'f} \\ Kj & \xrightarrow[\eta_j]{} & K'j \end{array} ,$$

under F, so by functoriality of F, it commutes and by $f : i \to j$ chosen arbitrarily, $F_*(\eta) : FK \Rightarrow FK'$. Now, consider $\mathrm{id}_K : K \Rightarrow K$ which is simply the collection of all identity morphism $(\mathrm{id}_K(j))_{j\in\mathcal{J}}$. It is clear that $F_*\mathrm{id}_K = (F\mathrm{id}_{K(j)})_j = (\mathrm{id}_{FK(j)})_j = \mathrm{id}_{FK}$ by functoriality of F. Lastly, given

$$K \overset{\eta}{\Rightarrow} L \overset{\varepsilon}{\Rightarrow} M \,,$$

in $\mathcal{C}^{\mathcal{J}}$, we have

$$F_*(\varepsilon\eta) = F_*((\varepsilon_j\eta_j)_j) = (F(\varepsilon_j\eta_j))_j = (F\varepsilon_j F\eta_j)_j = F_*\varepsilon F_*\eta \,,$$

by functoriality of F which completes the proof. Note that the same holds for G_* by $F, \mathcal{C}$ chosen arbitrarily. □

Lemma 1. *Given a natural transformation $\alpha : A \Rightarrow B$ between any functors $A, B : \mathcal{X} \to \mathcal{Y}$ we have the following natural transformation for any functor $A' : \mathcal{W} \to \mathcal{X}$:*

$$\alpha_{A'} := (\alpha_{A'w})_w : AA' \to BA' \,.$$

Proof. We know the diagram $\begin{array}{ccc} Ax & \xrightarrow{\alpha_x} & Bx \\ {\scriptstyle Af}\downarrow & & \downarrow{\scriptstyle Bf} \\ Ax' & \xrightarrow[\alpha_{x'}]{} & Bx' \end{array}$ commutes for all $f : x \to x'$ in $\mathcal{X}$, so of course we have

commutativity of $\begin{array}{ccc} AA'w & \xrightarrow{\alpha_{A'w}} & BA'w \\ {\scriptstyle AA'g}\downarrow & & \downarrow{\scriptstyle BA'g} \\ AA'w' & \xrightarrow[\alpha_{A'w'}]{} & BA'w' \end{array}$ for arbitrary $g : w \to w'$ in $\mathcal{W}$. □

Observation 1. *For all $K \in \mathcal{C}^{\mathcal{J}}$ and $L \in \mathcal{D}^{\mathcal{J}}$, we have maps*

$$\begin{aligned} \psi_{KL} : \mathcal{D}^{\mathcal{J}}(F_*K, L) &\longrightarrow \mathcal{C}^{\mathcal{J}}(K, G_*L) \\ \beta &\longmapsto G_*(\beta)\eta_K := (G(\beta_j)\eta_{Kj})_{j\in\mathcal{J}} \;, \\ \varphi_{KL} : \mathcal{C}^{\mathcal{J}}(K, G_*L) &\longrightarrow \mathcal{D}^{\mathcal{J}}(F_*K, L) \\ \alpha &\longmapsto \varepsilon_L F_*(\alpha) := (\varepsilon_{Lj}F(\alpha_j))_{j\in\mathcal{J}} \;. \end{aligned}$$

Proof. First we will show that ψ_{KL} is defined over all of $\mathcal{D}^{\mathcal{J}}(F_*K, L)$, i.e, given $\alpha \in \mathcal{D}^{\mathcal{J}}(F_*K, L)$ we will show that $G(\alpha)\eta_K$ is indeed in $\mathcal{C}^{\mathcal{J}}(K, G_*L)$. By our investigation in Remark 1, we have that $G_*(\alpha) : GFK \Rightarrow G_*L$ and by Lemma 1, we have $\eta_K : K = \mathrm{id}_{\mathcal{C}}K \Rightarrow GFK$, therefore, the composition $G_*(\alpha)\eta_K$ is a natural transformation $K \Rightarrow GL$. The same argument holds for the definedness of φ_{KL}, our previous results ensure $F_*(\beta) : FK \Rightarrow FGL$ and $\varepsilon_L : FGL \Rightarrow L$, so the composition $\varepsilon_L F_*(\beta)$ makes sense as a natural transformation $FK \Rightarrow L$. □

Observation 2. *For all K and L, ψ_{KL} is an isomorphism of sets (or proper classes) with inverse φ_{KL}.*

Proof. Consider arbitrary $\beta \in \mathcal{D}^{\mathcal{J}}(F_*K, L)$. Then $\varphi_{KL}(\psi_{KL}(\beta)) = (\varepsilon_{Lj}(FG\alpha_j)F\eta_{Kj})_{j\in\mathcal{J}}$ which is a natural transformation $FK \Rightarrow L$. Visually, we have commutativity of the following for all $j \in \mathcal{J}$:

$$FKj \xrightarrow{F\eta_{Kj}} FGFKj \xrightarrow{FG\beta_j} FGLj \xrightarrow{\varepsilon_{Lj}} Lj, \qquad FKj \xrightarrow{\varphi_{KL}(\psi_{KL}(\beta))_j} Lj$$

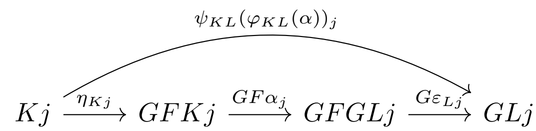
Then by the first triangle-inequality, and naturality of ε, we have commutativity of

$$\begin{array}{ccc} FKj & \xrightarrow{F\eta_{Kj}} & FGFKj \\ & {\scriptstyle \mathrm{id}_{FKj}}\searrow & \downarrow{\scriptstyle \varepsilon_{FKj}} \\ & & FKj \end{array} \quad \text{, and} \quad \begin{array}{ccccc} FGFKj & \xrightarrow{FG\beta_j} & FGLj & \xrightarrow{\varepsilon_{Lj}} & Lj \\ {\scriptstyle \varepsilon_{FKj}}\downarrow & & & \nearrow{\scriptstyle \beta_j} & \\ FKj & & & & \end{array} \quad , \tag{1}$$

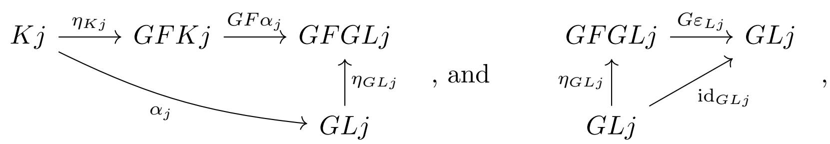
respectfully. Putting this together gives commutativity of

$$\begin{array}{ccc} FKj & \xrightarrow{\varphi_{KL}(\psi_{KL}(\beta))_j} & Lj \\ & {\scriptstyle \mathrm{id}_{FKj}}\searrow \quad \nearrow{\scriptstyle \beta_j} & \\ & FKj & \end{array} \; .$$

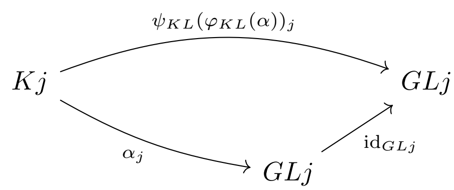
Similarly, given $\alpha \in \mathcal{C}^{\mathcal{J}}(K, G_*L)$, we have $\psi_{KL}(\varphi_{KL}(\alpha)) = (G\varepsilon_{Lj}(GF\alpha_j)\eta_{Kj})_{j\in\mathcal{J}}$. Fixing some $j \in \mathcal{J}$ we can visualize this natural transformation as



Then by naturality of η, and the second triangle-inequality, we have commutativity of

 (2)

respectfully. Putting this together gives commutativity of



□

Proposition 1. *The collection of isomorphisms* $\psi := (\psi_{KL})_{KL}$ *is natural in both variables* ($K \in (\mathcal{C}^{\mathcal{J}})\mathrm{op}$, $L \in \mathcal{D}^{\mathcal{J}}$)*, thus, it is a natural isomorphism* $\mathcal{D}^{\mathcal{J}}(F_*(-), -) \Rightarrow \mathcal{C}^{\mathcal{J}}(-, G_*(-))$.

Proof. To show naturality in the variable K, let us fix $L \in \mathcal{D}^{\mathcal{J}}$ and a natural transformation (morphism) $\alpha : K \Rightarrow K'$ in $\mathcal{C}^{\mathcal{J}}$. Observe that

$$\mathcal{D}^{\mathcal{J}}(F_*(-), L)(\alpha) := F_*(\alpha)^* : \Big(\beta : FK' \Rightarrow L\Big) \mapsto \Big(\beta F_*(\alpha) : FK \Rightarrow L\Big),$$

and

$$\mathcal{C}^{\mathcal{J}}((-), G_*L)(\alpha) := \alpha^* : \Big(\beta' : K' \Rightarrow GL\Big) \mapsto \Big(\beta'\alpha : K \Rightarrow GL\Big),$$

Now for any such $\beta \in \mathcal{D}^{\mathcal{J}}(F_*K', L)$, we have

$$\psi_{K'L}((F_*\alpha)^*(\beta)) = \psi_{K'L}(\beta F_*\alpha) = G_*\beta(G_*F_*\alpha)\eta_K\,.$$

Analogous (and identical after replacing GL with K') to commutativity of the first diagram in (2), naturality of η gives us $(G_*F_*\alpha)\eta_K = \eta_{K'}\alpha$. Therefore,

$$\psi_{K'L}((F_*\alpha)^*(\beta)) = G_*\beta\eta_{K'}\alpha\,.$$

Additionally,

$$\alpha^*(\psi_{KL}(\beta)) = \alpha^*(G_*\beta\eta_{K'}) = G_*\beta\eta_{K'}\alpha\,.$$

That is, we have commutativity of the naturality diagram for the variable K:

$$\begin{array}{ccc} \mathcal{D}^{\mathcal{J}}(FK', L) & \xrightarrow{\psi_{K'L}} & \mathcal{C}^{\mathcal{J}}(K', GL) \\ {\scriptstyle (F_*\alpha)^*}\uparrow & & \uparrow{\scriptstyle \alpha^*} \\ \mathcal{D}^{\mathcal{J}}(FK, L) & \xrightarrow{\psi_{KL}} & \mathcal{C}^{\mathcal{J}}(K, GL) \end{array}$$

Similarly, for fixed $K \in \mathcal{C}^{\mathcal{J}}$, $\beta : F_*K \Rightarrow L$, and $\gamma : L \Rightarrow L'$, we have

$$\psi_{KL'}(\gamma_*(\beta)) = \psi_{KL'}(\gamma\beta) = G_*\gamma G_*\beta\eta_K$$

and

$$(G_*\gamma)_*\psi_{KL}(\beta) = G_*\gamma G_*\beta\eta_K \,,$$

so the diagram for naturality in L also commutes:

$$\begin{array}{ccc} \mathcal{D}^{\mathcal{J}}(FK, L) & \xrightarrow{\psi_{KL}} & \mathcal{C}^{\mathcal{J}}(K, GL) \\ {\scriptstyle \gamma_*}\downarrow & & \downarrow{\scriptstyle (G_*\gamma)_*} \\ \mathcal{D}^{\mathcal{J}}(FK, L') & \xrightarrow{\psi_{KL'}} & \mathcal{C}^{\mathcal{J}}(K, GL') \end{array}$$

□

In hindsight, an easier way to prove all of this would have been from the following ansatz for unit and counit, respectfully:

$$\widetilde{\eta} := (\eta_K)_K : \mathrm{id}_{\mathcal{C}^{\mathcal{J}}} \Rightarrow G_*F_* \,, \quad \widetilde{\varepsilon} := (\varepsilon_L)_L : F_*G_* \Rightarrow \mathrm{id}_{\mathcal{D}^{\mathcal{J}}} \,.$$

Nevertheless, we arrive at the initially sought-after statement (and a little extra):

Corollary 1. *For all $C \in \mathcal{C}$ and functors $K : \mathcal{J} \to \mathcal{D}$, there is an iso*

$$\mathrm{Cone}(F\Delta C, K) \cong \mathrm{Cone}(\Delta C, GK) \,.$$

Furthermore, the collection of such isos is natural in the variables C and K.

Proof. The existence of such an iso follows directly from the identification:

$$\mathrm{Cone}(F\Delta C, K) := \mathcal{D}^{\mathcal{J}}(F\Delta C, K) \,, \quad \mathrm{Cone}(\Delta C, GK) := \mathcal{C}^{\mathcal{J}}(\Delta C, GK) \,.$$

We then know, from Observation 2 that our previously constructed $\psi_{\Delta CK}$ gives the desired isomorphism. Furthermore, Proposition 1 gives naturality in the variables $\Delta C, K$ and this pulls back to naturality in C by functoriality of $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$. In particular a morphism $f : C \to C'$ in $\mathcal{C}$ induces a natural transformation $\Delta f : \Delta C \Rightarrow \Delta C'$ over which this collection of isos is natural. □

Solution 2

1. Given terminal objects T and T' in $\mathcal{C}$, we necessarily have unique isomorphisms $t : T \to T'$ and $t' : T' \to T$. Now considering some $f : C \to T$, we can compose $t'tf : C \to T$ and then by uniqueness of the morphisms $C \to T$, we have $t'tf = f$. The same argument yields $tt'g = g$ for $g : C \to T'$. We can then choose $f = \mathrm{id}_T$, $g = \mathrm{id}_{T'}$ and we see

$$t't = t't\mathrm{id}_T = \mathrm{id}_T \,, \quad tt' = tt'\mathrm{id}_{T'} = \mathrm{id}_{T'} \,.$$

□

2. By our second definition of limit cone (as terminal objects in $\int_{\mathcal{C}} \mathrm{Cone}(-, F)$), the result is clear. If $\alpha : \Delta L \Rightarrow F$ and $\alpha' : \Delta L' \Rightarrow F$ are two limit cones, then (L, α) and (L', α) are both terminal objects in $\int_{\mathcal{C}} \mathrm{Cone}(-, F)$ admitting an iso $i : (L, \alpha) \to (L', \alpha')$ in $\int_{\mathcal{C}} \mathrm{Cone}(-, F)$. Recall that i is necessarily a morphism $L \to L'$ in $\mathcal{C}$ such that $\mathrm{Cone}(-, F)i = \Delta i = i$ (viewed as a natural transformation $\Delta L \Rightarrow \Delta L'$) satisfies $\alpha' i = \alpha$. The important part here is that i is an iso, so $i^{-1} i = \mathrm{id}_{(L,\alpha)} = \mathrm{id}_L$ by definition and similarly $ii^{-1} = \mathrm{id}_{(L',\alpha')} = \mathrm{id}_{L'}$. In other words, isomorphisms in category of elements always descends to an isomorphism in the original category. □

Solution 3

Consider the cone $\varepsilon : \Delta FI \Rightarrow F$ where $\varepsilon_j = Fe_j$, the image of the unique $e_j : I \to j$ in J. This is indeed a cone because for all $f : j \to j'$ in J, we of course have commutativity of

$$\begin{array}{ccc} (\Delta FI)j = FI & \xrightarrow{\varepsilon_j = Fe_j} & Fj \\ {\scriptstyle (\Delta FI)f = \mathrm{id}_{FI}}\big\downarrow & & \big\downarrow{\scriptstyle Ff} \\ (\Delta FI)j' = FI & \xrightarrow[\varepsilon_{j'} = Fe_{j'}]{} & Fj' \end{array} ,$$

because $FfFe_j = F(fe_j) = Fe_{j'}$ by uniqueness of the map $I \to j'$. We will now show that $\varepsilon : \Delta FI \Rightarrow F$ is the *limit* cone. For all other cones $\alpha : \Delta C \Rightarrow F$, let us define $\pi_\alpha := \alpha_I$. Observe that for all $f : j \to j'$ in J, our previous result $FfFe_j = Fe_{j'}$, along with naturality of α implies commutativity of

$$\begin{tikzcd} \Delta C \arrow[rrd, bend left, "\alpha_{j'}"] \arrow[rd, "\alpha_I"] \arrow[rdd, bend right, "\alpha_j"'] & & \\ & FI \arrow[r, "Fe_{j'}"] \arrow[d, "Fe_j"'] & Fj' \\ & Fj \arrow[ru, "Ff"'] & \end{tikzcd} ,$$

which, by $f : j \to j'$ chosen arbitrarily, gives commutativity of

$$\begin{tikzcd} \Delta C \arrow[d, Rightarrow, "\pi_\alpha := \alpha_I"'] \arrow[rd, Rightarrow, "\alpha"] & \\ \Delta FI \arrow[r, Rightarrow, "\varepsilon"'] & F \end{tikzcd} .$$

We have thus shown the existence of a morphism $\pi_\alpha : (C, \alpha) \to (FI, \varepsilon)$ in $\int_{\mathcal{C}} \mathrm{Cone}(-, F)$ for all (C, α). It remains to show uniqueness to declare that (FI, ε) is terminal in $\int_{\mathcal{C}} \mathrm{Cone}(-, F)$ and therefore, $\varepsilon : \Delta FI \Rightarrow F$ is the limit cone (by imposing the solution to the previous problem). Consider some morphism $f : C \to FI$ (which trivially induces $f : \Delta C \Rightarrow \Delta FI$) such that

$$\begin{tikzcd} \Delta C \arrow[d, Rightarrow, "f"'] \arrow[rd, Rightarrow, "\alpha"] & \\ \Delta FI \arrow[r, Rightarrow, "\varepsilon"'] & F \end{tikzcd} ,$$

commutes. Then in particular, we have commutativity of

$$\begin{tikzcd} C \arrow[d, "f"'] \arrow[rd, "\alpha_I"] & \\ FI \arrow[r, "\varepsilon_I = Fe_i = F\mathrm{id}_I = \mathrm{id}_{FI}"'] & FI \end{tikzcd} ,$$

which decisively settles uniqueness. □

Solution 4

1. Cones (cocones) over (under) these diagrams are collections of morphisms in the poset. Therefore, there is a cone over F with summit P if and only if $P \leq Fj$ for all $j \in J$, and there is a cocone under F with nadir P if and only if $Fj \leq P$ for all $j \in J$. It is then clear that the limit of F is the infimum of $\mathrm{im}F$ and the colimit of F is the supremum of $\mathrm{im}F$.

2. We will first prove that for an arbitrary finite number n, the product of $V_1, ..., V_n$ is simply the direct product $V_1 \oplus ... \oplus V_n$. Recall that we have projection and inclusion maps

$$\begin{aligned} \pi_j &: v_1 + ... + v_n \longmapsto v_j \,, \\ \mathrm{i}_j &: v_j \longmapsto 0_{V_1} + ... + 0_{V_{j-1}} + v_j + 0_{V_{j+1}} + ... + 0_{V_n} \,, \end{aligned}$$

whenever $v_1 \in V_1, ..., v_n \in V_n$, i.e., $v_1 + ... + v_n \in V_1 \oplus ... \oplus V_n$. A crucial fact is that $\pi_j \mathrm{i}_j = \mathrm{id}_{V_j}$ for all j and $\pi_j \mathrm{i}_k = 0_k j : V_k \to 0_{V_j}$ for all $j \neq k$. Now for any cone $(f_i : W \to V_i)_i$, we can define

$$\begin{aligned} f : W &\longrightarrow V_1 \oplus ... \oplus V_n \\ w &\longmapsto \mathrm{i}_1 f_1(w) + ... + \mathrm{i}_n f_n(w) \,, \end{aligned}$$

(for which linearity is clear) and we see

$$\pi_j f(w) = \pi_j \mathrm{i}_j f_j(w) = \mathrm{id}_{V_j} f_j(w) \tag{3}$$

for all j. Further, if there was another $f' : W \to V_1 \oplus ... \oplus V_n$ satisfying (3) for all j then we would have $f = f'$. This is because for all $v \in V_1 \oplus ... \oplus V_n$, this vector is fully characterized by its image under the projection maps $(\pi_j)_j$ implying that for all $w \in W$, $f(w)$ and $f'(w)$ are fully characterized by their images under the projection maps $(\pi_j)_j$ which are necessarily equal by assumption. Then imposing the universal property yields the result.

□

Next, we will prove that $V_1 \oplus ... \oplus V_n$ is also the coproduct of $V_1, ..., V_n$. Observe that for any cocone $(g_i : V_i \to W)_i$, we can define

$$\begin{aligned} g : V_1 \oplus ... \oplus V_n &\longrightarrow W \\ v &\longmapsto g_1 \pi_1(v) + ... + g_n \pi_n(v) \,, \end{aligned}$$

(again, linearity is clear) and we see that for all $v_j \in V_j$,

$$\begin{aligned} g\mathrm{i}_j(v_j) &= g_1 \pi_1 \mathrm{i}_j(v_j) + ... + g_j \pi_j \mathrm{i}_j(v_j) + ... + g_n \pi_n \mathrm{i}_j(v_j) \\ &= g_1(0_{V_1}) + ... + g_j(v_j) + ... + g_n(0_{V_n}) \\ &= 0_W + ... + g_j(v_j) + ... + 0_W \\ &= g_j(v_j) \end{aligned}$$

Now uniqueness of a map satisfying this property, $g\mathrm{i}_j v_j = g_j v_j$ for all $v_j \in V_j$, is clear because if we had another such map g', we could evaluate it on arbitrary $v = v_1 + ... + v_n \in V_1 \oplus ... \oplus V_n$ and obtain (by linearity of g and g')

$$g'(v) = \sum_j g' \mathrm{i}_j(v_j) = \sum_j g_j(v_j) = \sum_j g\mathrm{i}_j(v_j) = g(v)$$

Again, the corresponding universal property will yield the result.

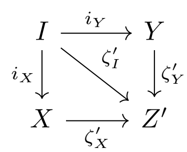
□

We remark that these general n cases recover the original statement of investigation: “Prove that in the category Vect, both the product and the coproduct of two vector spaces V_1, V_2 are given by their direct sum.” Now turning to the case of infinite products and coproducts, we see that this truth is not retained. For a counterexample consider $\{\Bbbk[x^j]\}_{j\in\mathbb{N}}$ as our basic vector spaces. If $\oplus_j \Bbbk[x^j]$ was our product, then we would have a map $f : \Bbbk[[x]] \to \oplus_j \Bbbk[x^j]$ (where $\Bbbk[[x]]$ vector space of formal power series over $\Bbbk$) such that for all $v \in \Bbbk[[x]]$ and $j \in \mathbb{N}$, $\pi_j f(v) = \widetilde{\pi}_j(v)$ (where $\pi_j, \widetilde{\pi}_j$ are the projections $\oplus_k \Bbbk[x^k] \to \Bbbk[x^j]$, $\Bbbk[[x]] \to \Bbbk[x^j]$, respectfully). However, this cannot be the case because we can consider $v = \sum_j x^j$. Necessarily, $f : \sum_j x^j \mapsto \sum_j \mathrm{i}_j(x^j)$ (where i_j is the inclusion $\Bbbk[x^j] \hookrightarrow \oplus_k \Bbbk[x^k]$) which is an infinite sum and thus, not actually an element of $\oplus_k \Bbbk[x^k]$.

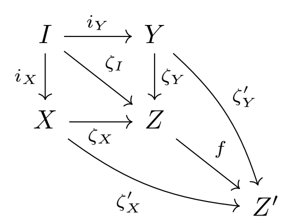
□

Solution 5

1. Let us use Z to denote the pushout of the given diagram with colimit cocone ζ, and Z' to denote the coproduct of X and Y with colimit cocone ζ'. Further, for $C \in \mathcal{C}$ we will use i_C do denote the unique morphism $I \to C$. Note that we can treat ζ' as a cocone under $X \xleftarrow{i_X} I \xrightarrow{i_Y} Y$ by adding an extra leg $\zeta'_I := i_{Z'}$. To see this, observe that the diagram

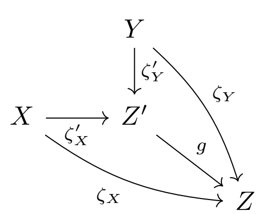


will necessarily commute by uniqueness of the morphism $I \to Z'$. Therefore, by the fact that Z is the pushout, we have a unique morphism $f : Z \to Z'$ such that



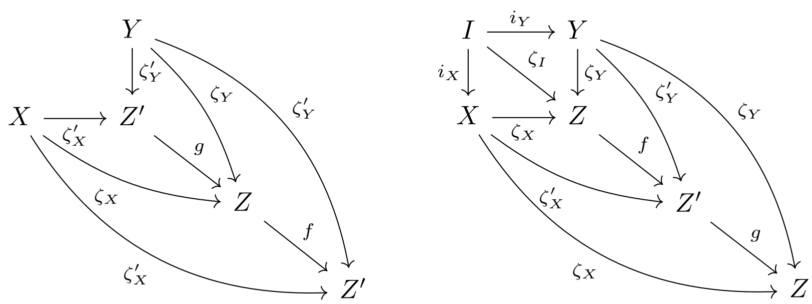
(4)

commutes. Similarly we can treat ζ as a cocone under the discrete diagram $X \quad Y$ (after discarding ζ_I) and this admits a unique morphism $g : Z' \to Z$ such that the diagram



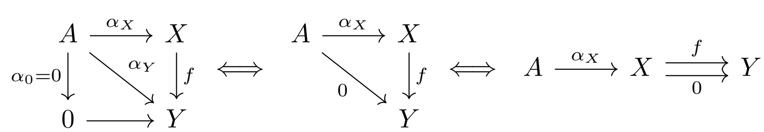
(5)

commutes. Combining the diagrams in (4) and (5) yields commutativity of both diagrams below.

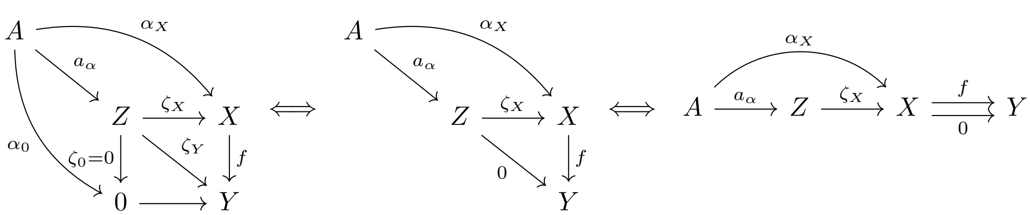


Note that the diagram on the right carries the additional information that $f\zeta_I = i_{Z'} = \zeta'_I$ and $gf\zeta_I = i_Z = \zeta_I$ and each "layer" (if you will) of both diagrams is a cocone. We then impose the universal property for the respective colimits (coproduct for left, pushout for right) stating that fg and gf are the unique morphisms making the relevant legs commute. Thus, $fg = \mathrm{id}_{Z'}$, $gf = \mathrm{id}_Z$, and we have $Z \cong Z'$.

2. We will proceed a bit different than before. Instead of giving an isomorphism between the two objects, we will show that a pullback satisfies the universal property for the equalizer and vice-versa. Note that any cone over the first diagram is necessarily a cone over the second diagram and vice-versa.To see this, observe that we have equivalence of the commutativity of the diagrams below.

 (6)

This equivalence is given by the uniqueness of maps that factor through the zero object, in particular, $(0\alpha_X : A \to X \to 0 \to Y) = (0 : A \to 0 \to Y)$. We can also insert a second cone (ζ with summit Z) and an arbitrary morphism $a_\alpha : A \to Z$ into the picture to get a stronger equivalence of commutativity:

 (7)

Now the result will follow effortlessly. Let Z be the pushout of the first diagram and ζ its limit cone. To prove that it is the equalizer of the second diagram consider a cone

$$A \xrightarrow{\alpha_X} X \underset{0}{\overset{f}{\rightrightarrows}} Y \ .$$

By (6), we know that this induces a cone over the first diagram admitting (by universal property for the pullback Z) the unique map a_α making the first diagram in (7) commute. Then following the equivalence of (7) to the right we get commutativity of the rightmost diagram, that is we get existence of a map (in this case a_α) making the rightmost diagram commute. For uniqueness, suppose we had two such maps a_α, a'_α making the rightmost diagram of (7) commute. Then following (7) leftward, we get that these maps both force the leftmost diagram in (7) to commute and then by universal property for the pullback, Z, we have that the two maps are equal. Thus, Z is also an equalizer of the second diagram.

Analogously, we can prove that the equalizer of the second diagram is a pullback of the first. Let Z be the equalizer of the second diagram and ζ its limit cone. To prove that it is the pullback of the first diagram consider a cone

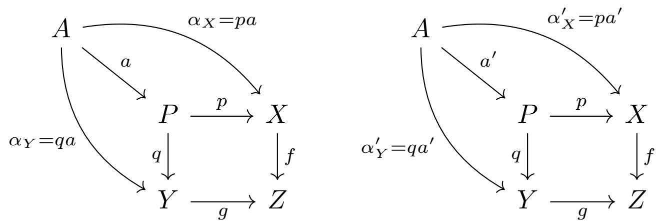
$$\begin{array}{ccc} A & \xrightarrow{\alpha_X} & X \\ {\scriptstyle \alpha_0=0}\downarrow & \searrow^{\alpha_Y} & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & Y \end{array} .$$

By (6), we know that this induces a cone over the second diagram admitting (by universal property for the equalizer Z) the unique map a_α making the second diagram in (7) commute. Then following the equivalence of (7) to the left we get commutativity of the leftmost diagram, that is we get existence of a map (in this case a_α) making the leftmost diagram commute. For uniqueness, suppose we had two such maps a_α, a'_α making the leftmost diagram of (7) commute. Then following (7) rightward, we get that these maps both force the rightmost diagram in (7) to commute and then by universal property for the equalizer, Z, we have that the two maps are equal. Thus, Z is also a pullback of the first diagram.

□

Solution 6

Suppose f is a monomorphism and consider arbitrary $a, a' : A \to P$ which we can extend to unique cones α, α' over the diagram $Y \xrightarrow{g} Z \xleftarrow{f} X$ by the assumption that P is the pullback:



Now to show that q is mono, let us assume that $qa = qa'$, i.e., $\alpha_Y = \alpha'_Y$. If this is the case, then obviously $\alpha_Z = gqa = gqa' = \alpha'_Z$, but more interestingly,

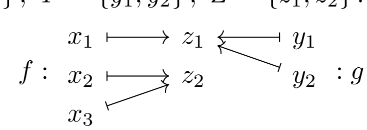
$$fpa = gqa = gqa' = fpa' ,$$

so by f mono, we have $pa = pa'$ and thus, $\alpha_X = \alpha'_X$. Therefore, our two cones α, α' are equal and by the universal property of the pushout P, we necessarily have $a = a'$.

□

For a counterexample of the converse, Let us consider one of the most simple example possible in the category Set:

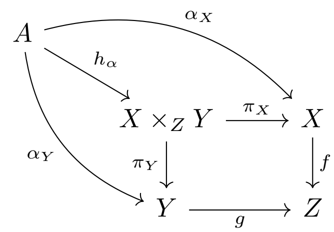
$$X = \{x_1, x_2, x_3\} , \; Y = \{y_1, y_2\} , \; Z = \{z_1, z_2\} .$$



In Set, the pullback is the fiber product

$$P = X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\} = \{(x_1, y_1), (x_1, y_2)\} ,$$

and the (nontrivial) legs of the limit cone are the projection maps $p = \pi_X$ and $q = \pi_Y$. It is then clear to see that f is not mono (injective in Set) yet q is. It remains to prove that the fiber product is indeed the pullback in Set. To see this, let's again choose $Y \xrightarrow{g} Z \xleftarrow{f} X$ to be arbitrary in Set and consider some cone α over this diagram with summit A. Then necessarily, $g\alpha_Y(a) = f\alpha_X(a) = \alpha_Z(a)$ for all $a \in A$ and we can construct $h_\alpha : A \to X \times_Z Y$ (fiber product) by $h_\alpha : a \mapsto (\alpha_X(a), \alpha_Y(a))$ making the following diagram commute:

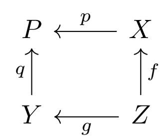


Uniqueness of such an h_α is clear because if we had h'_α also making the above diagram commute, then $h'_\alpha(a) = (x, y)$ such that $\alpha_X(a) = \pi_X(x, y) = x$ and $\alpha_Y(a) = \pi_Y(x, y) = y$.

□

The dual statement to the one above is the following:

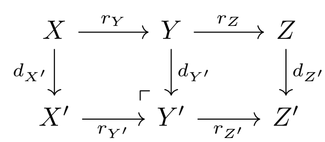
"In a pushout square



if f is an epimorphism then so is q."

Solution 7

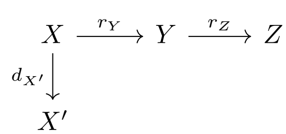
Before we do anything, let's put some names to these morphisms (r for right, d for down, subscript for codomain):



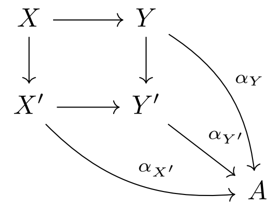
Note that when a morphism is not specified, it is assumed to be the one in the diagram above with the corresponding domain and codomain.

$(\implies)$

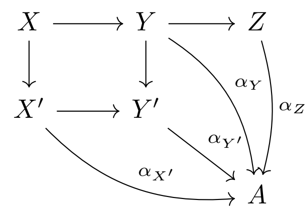
Now suppose the right-hand square is a pushout. Then given a cocone α with nadir A under $X' \xleftarrow{d_{X'}} X \xrightarrow{r_Z r_Y} Z$, we can first extend this to a cocone under



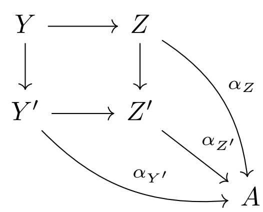
by simply adding a leg $\alpha_Y := \alpha_Z r_Z$. Then we can restrict this enhanced cocone to the cocone $(\alpha_{X'}, \alpha_X, \alpha_Y)$ and, by the assumption that Y' is a pushout in this left-handed square, there exists a unique $\alpha_{Y'}$ inducing commutativity of

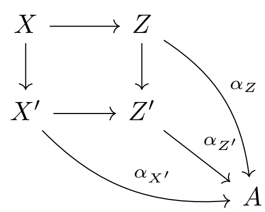


Recalling that $\alpha_Y := \alpha_Z r_Z$, we can extend this commutativity to that of



Restricting once again, this time to the cocone $(\alpha_{Y'}, \alpha_Y, \alpha_Z)$, and imposing the universal property of Z' gives us a unique morphism $\alpha_{Z'}$ inducing commutativity of the below left diagram which, in combination with the commutativity of the previous diagram, implies that for the below right diagram.





To clarify uniqueness of such an $\alpha_{Z'}$, let us recall the uniqueness in each step:

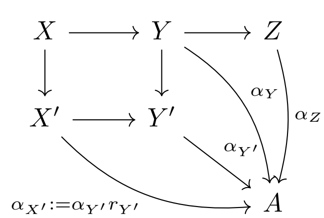
$$\alpha = (\alpha_{X'}, \alpha_X, \alpha_Z) \rightsquigarrow \text{ unique } (\alpha_{X'}, \alpha_X, \alpha_Y) \rightsquigarrow \text{ unique } \alpha_{Y'} \rightsquigarrow \text{ unique } (\alpha_{Y'}, \alpha_Y, \alpha_Z) \rightsquigarrow \text{ unique } \alpha_{Z'}\,.$$

Therefore, Z' satisfies the universal property for the pushout in the composite square.

□

($\Longleftarrow$)

Suppose that the composite square is a pushout and consider an arbitrary cocone α with nadir A under $Y' \longleftarrow Y \longrightarrow Z$. Then we can uniquely extend α to the following cocone:

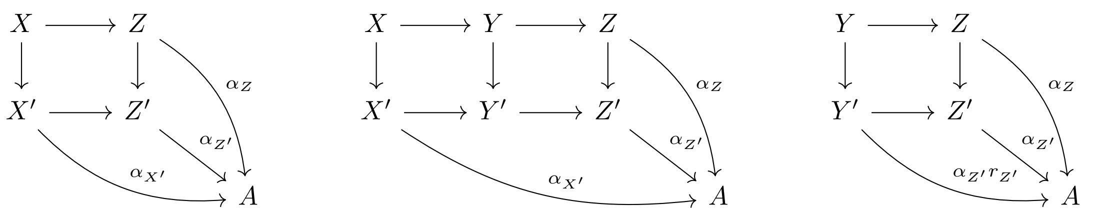


(8)

Now, this extended cocone restricts uniquely[1] to the cocone $(\alpha_{X'}, \alpha_X, \alpha_Z)$ for which there is a unique $\alpha_{Z'}$ inducing commutativity of the below left diagram (by the assumption that the composite square is a pushout). From this, we observe that the composite square factors uniquely through the left-hand square

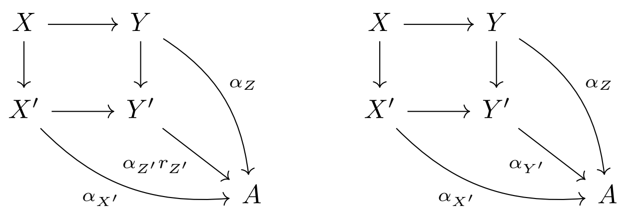
[1] All restrictions are unique and this holds not just for cones, but for pretty much any mathematical object by definition.

(by our pushout assumption on the left hand square) giving commutativity of the below middle diagram. Finally, this yields commutativity of the below right diagram.



(9)

We must now check that $\alpha_{Z'} r_{Z'} = \alpha_Y$. This is true because the commutativity of the above middle diagram and that of the diagram in (8) restricts to commutativity of the below left and below right diagrams, respectively.



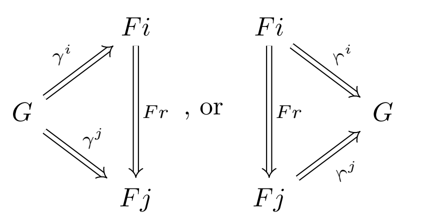
Therefore, by universal property of the pushout Y', we have $\alpha_{Z'} r_{Z'} = \alpha_{Y'}$. Just is we did in the proof of the right implication, we will clarify that uniqueness was maintained throughout this construction to conclude that Z' satisfies the universal property of the pushout in the right-hand diagram:

$$\alpha = (\alpha_{Y'}, \alpha_Y, \alpha_Z) \rightsquigarrow \text{ unique } (\alpha_{X'}, \alpha_X, \alpha_{Y'}, \alpha_Y, \alpha_Z) \rightsquigarrow \text{ unique } (\alpha_{X'}, \alpha_X, \alpha_Z) \rightsquigarrow \text{ unique } \alpha_{Z'}\,.$$

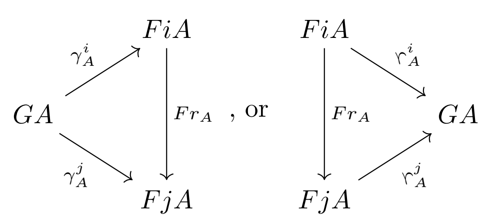
□

Solution 8

Let's begin by recalling what a cone with summit (or cocone with nadir) G over (under) $F : J \to \mathcal{C}^{\mathcal{A}}$ would look like. This is a collection of morphisms in $\mathcal{C}^{\mathcal{A}}$, i.e., natural transformations, $\gamma := (\gamma^j : G \Rightarrow Fj)_{j \in J}$ (or $(\curlyvee^j : Fj \Rightarrow G)_{j \in J}$) satisfying the "pairwise commutativity" property that for all $r : i \to j$ in J, the following diagram



commutes. In particular this is equivalent to the requirement for all $r : i \to j$, and $A \in \mathcal{A}$, the diagram



(10)

commutes (in $\mathcal{C}$). This equivalence holds because the composition of natural transformations $(\eta_A)_A : T \Rightarrow T'$ and $(\eta'_A)_A : T' \Rightarrow T''$ is given, object-wise, by $(\eta'_A \eta_A)_A$. An important observation is that γ (or $\rotatebox[origin=c]{180}{γ}$) restricts to the collection $\gamma_A := (\gamma^j_A)_j$ (or $\rotatebox[origin=c]{180}{γ}_A := (\rotatebox[origin=c]{180}{γ}^j_A)_j$) which will then form a cone (or cocone) over (or under) $\mathrm{ev}_A F$ for all fixed A. This is because a remark to the above formulation of γ as a cone tells us that for all $r : i \to j$ and *fixed* A, we still have commutativity of Diagram (10). We will call γ_A (or $\rotatebox[origin=c]{180}{γ}_A$) the A-slice of the ambient cone $\gamma = ((\gamma^j_A)_A)_j$ (or ambient cocone $\rotatebox[origin=c]{180}{γ} = ((\rotatebox[origin=c]{180}{γ}^j_A)_A)_j$).

Now suppose that the limits of the diagrams

$$J \xrightarrow{F} \mathcal{C}^{\mathcal{A}} \xrightarrow{\mathrm{ev}_A} \mathcal{C}$$

exist in $\mathcal{C}$ for all $A \in \mathcal{A}$. Let us denote these limits by $\lim \mathrm{ev}_A F$ and their limit cones as $\alpha_A := (\alpha^j_A)_{j \in J}$ for each $A \in \mathcal{A}$. This implies that all sliced cones $(\gamma^j_A)_j$ will factor uniquely through $\lim \mathrm{ev}_A F$ as described by the commutative diagram below.

$$\begin{array}{ccc} & & \mathrm{ev}_A F \\ & \overset{\gamma_A}{\nearrow\!\!\!\nearrow} & \overset{\alpha_A}{\Uparrow} \\ GA & \xrightarrow{h_{\gamma_A}} & \lim \mathrm{ev}_A F \end{array} \tag{11}$$

Now the above diagram lives in $\mathcal{C}$ and we would like to get the analogous factoring result for the ambient cone γ in $\mathcal{C}^{\mathcal{A}}$. For this, let us define the functor

$$\begin{aligned} \lim \mathrm{ev}_{(-)} F : \mathcal{A} &\longrightarrow \mathcal{C} \\ A &\longmapsto \lim \mathrm{ev}_A F \\ (f : A \to A') &\longmapsto f_\alpha \end{aligned}$$

where f_α is the unique morphism $\lim \mathrm{ev}_A F \to \lim \mathrm{ev}_{A'} F$ inducing commutativity of the following diagram for all $r : i \to j$ in J:

$$\begin{array}{ccccc} & \overset{\alpha^i_A}{\nearrow} & FiA & \xrightarrow{Fif} & FiA' \\ & & & \overset{\alpha^i_{A'}}{\nearrow} & \downarrow{\scriptstyle Fr_{A'}} \\ \lim \mathrm{ev}_A F & \dashrightarrow & \lim \mathrm{ev}_{A'} F & & \\ & & & \underset{\alpha^j_{A'}}{\searrow} & \\ & \underset{\alpha^j_A}{\searrow} & FjA & \xrightarrow{Fjf} & FjA' \end{array}$$

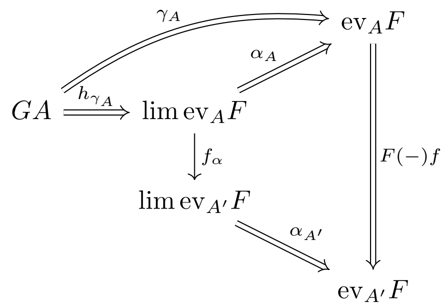
Note that existence and uniqueness of f_α is given by the fact that $((Fjf)\alpha^j_A)_j$ is a cone over $\mathrm{ev}_{A'} F$, so it factors uniquely through $\lim \mathrm{ev}_{A'} F$. *Aside:* There is a more elegant way of defining f_α by the unique morphism in $\mathcal{C}$ inducing commutativity of

$$\begin{array}{ccccc} & & \mathrm{ev}_A F & \overset{F(-)f}{\Longrightarrow} & \mathrm{ev}_{A'} F \\ & \overset{\alpha_A}{\nearrow\!\!\!\nearrow} & & \underset{\alpha_{A'}}{\nearrow\!\!\!\nearrow} & \\ \Delta_J \lim \mathrm{ev}_A F & & \dashrightarrow & & \Delta_J \lim \mathrm{ev}_{A'} F \end{array} \tag{12}$$

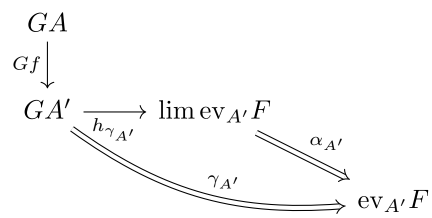
where $\Delta_J C : J \to \mathcal{C}$ by sending all objects to C and all morphisms to id_C, and $F(-)f := (Fjf)_{j \in J}$ (yes, we should denote $F(-)f$ as $\mathrm{ev}_{(-)} F$ which is a functor that is basically the same as F and whose limit, wrt the J slot, is the functor $\lim \mathrm{ev}_{(-)} F$, but let's keep it simple and pretend that their is no higher magic happening here). The fact that $\lim \mathrm{ev}_{(-)} F$ maps identities to identities is clear and composition preservation is also clear by concatenating the diagram above. Therefore, $\lim \mathrm{ev}_{(-)} F$ is indeed a functor.

We will now work to prove that $\lim \mathrm{ev}_{(-)} F = \lim F$ with limit cone $\alpha := ((\alpha^j_A)_A)_j : \Delta_J \lim \mathrm{ev}_{(-)} F \Rightarrow F$.

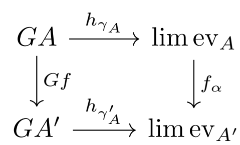
Claim: The collection $h_\gamma := (h_{\gamma_A})_A$, where h_{γ_A} is the unique morphism inducing commutativity in (11), is a natural transformation $G \Rightarrow \lim \mathrm{ev}_{(-)} F$. This is because for every $f : A \to A'$ in $\mathcal{A}$, the commutativity of Diagram (11) in combination with that of Diagram (12) gives commutativity of the diagram below



which, when combined with the A' version of Diagram (11), yields commutativity of the following:

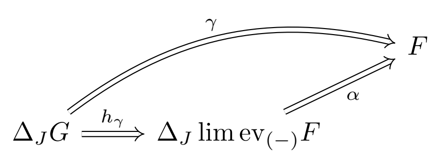


Now because γ is a cone with summit G, it is pretty clear that $\gamma_{A'} Gf = F(-)f\gamma_A$. Therefore, we have two equal cones with summit GA over $\mathrm{ev}_{A'} F$, thus they must factor through $\lim \mathrm{ev}_{A'} F$ uniquely, i.e., the diagram below must commute.



□

Next we can observe that h_γ induces commutativity of



because as explained in the beginning of the problem, it is sufficient to show that the restriction to Diagram (11) commutes for all choices of A. We conclude with the observation that h_γ is the unique natural transformation $G \Rightarrow \lim \mathrm{ev}_{(-)} F$ inducing commutativity of the above diagram because any such morphism would necessarily induce commutativity of the restricted (or "sliced") diagram (11)—where we have uniqueness—for all A. This concludes the proof that the functor $\lim \mathrm{ev}_{(-)} F$ is the limit $\lim F$. Finally, we remark that all our hard work as also proven the dual statement: "if the colimits of the diagram $\mathrm{ev}_A F$ exist in $\mathcal{C}$ for all $A \in \mathcal{A}$, then we can construct $\mathrm{colim}\, F$." The proof of this dual statement is simply the dual of the proof above.

□