# Clarifying The Tangent Space

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November, 2024

**Abstract**

As I young student, I found the notion of a tangent space (for a general manifold) to be incredibly difficult. While I am sure there are plenty of excellent expositions, I felt restricted to the extrinsic and coordinate dependent definitions. Upon mentoring a directed reading program following *Representations of Compact Lie Groups* by Bröcker and Dieck, it became clear that this supplementary material was necessary. The present work was written for my student with the assumed knowledge of manifolds and differentiable functions—only their definitions will be relevant.

## 1 The Tangent Bundle $TM$

### 1.1 $TM$ At A Point

Bröcker and Dieck define the tangent space using the notion of germs and derivations thereon (derivations being operators which satisfy the product rule). While it is quite neat that one can recover the intuitive notion of directional derivative from the abstraction derivations, I think for newcomers, we might as well start with the Tangent space *as* the vector space of directional derivatives. With this in mind, I would like to reintroduce this concept using only a manifold's atlas and differentiable paths into the manifold. We will see that the use of an equivalence relation shifts from the set of functions (yielding germs) to that of paths (yielding derivations).

Let $\Gamma_p M$ denote the set of differentiable paths $\gamma : (-\delta, \delta) \to M$ with $\gamma(0) = p$ (here $\delta > 0$) is arbitrary). Such a path naturally gives us a way to differentiate functions at the point $p$:

$$\begin{aligned} D_\gamma(-) : C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto D_\gamma f := \lim_{\varepsilon \to 0} \frac{f(\gamma(\varepsilon)) - f(p)}{\varepsilon} \end{aligned}$$

From here, we can define the **tangent space** at $p$ as $T_p M := \{D_\gamma \mid \gamma \in \Gamma_p M\}$.

*Remark* 1. *$T_p M$ is a vector space.*

*Proof.* Exercise. Hint: Prove it in $^n$, where one can add paths together, and use a local chart. □

*Remark* 2. If $U \subset M$ is some open subset and $p \in U$, then $T_p U = T_p M$.

I promised the reader that an equivalence class would arise and indeed this is the case: $T_p M \cong \Gamma_p M / \sim$ where $\gamma \sim \alpha$ if $D_\gamma = D_\alpha$. While this is somewhat underwhelming as it follows directly from the definition, we can give this isomorphism more character by promoting it from an equivalence in $\mathsf{Set}$ to an equivalence in $\mathsf{Sh}(\mathbb{R})$ (Sheaves over $\mathbb{R}$) which we will (maybe, but probably not) investigate in subsequent sections.

**Example 1** **($T_p\mathbb{R}^n$).** When our manifold is flat, we have $D_\gamma(-) = \gamma'(0) \cdot \nabla(-)$. Then, because the only $\gamma$-dependence of $D_\gamma(-)$ is through the vector $\gamma'(0)$, we see $D_\gamma = D_\alpha$ if and only if $\gamma'(0) = \alpha'(0)$, i.e., $T_p\mathbb{R}^n \cong \Gamma_p\mathbb{R}^n/\sim'$ where $\gamma \sim' \alpha$ if $\gamma'(0) = \alpha'(0)$. Further, $T_p\mathbb{R}^n \cong \mathbb{R}^n$ through the correspondence $D_\gamma(-) \mapsto \gamma'(0)$

In general setting, we don't *a priori* have any vector "tangent" to the path $\gamma$ at $p$ because our manifold need not be embedded in some flat ambient space. Recall that we do, however, have a local chart $\chi_U$ around $p$ through which we can push our paths:

$$\begin{aligned} \chi_U \circ - : \Gamma_p U &\longrightarrow \Gamma_p\mathbb{R}^n \\ \gamma &\longmapsto \chi_U \circ \gamma \end{aligned}$$

Further, we can define the **differential** of this chart,

$$\begin{aligned} (d\chi_U)_p : T_p U &\longrightarrow T_p\mathbb{R}^n \\ D_\gamma(-) &\longmapsto (\chi_U \circ \gamma)'(0) \cdot \nabla(-) \end{aligned}$$

Now recall that Remark 2 tells us we can lift these maps to the following:

$$\chi_U \circ - : \Gamma_p M \longrightarrow \Gamma_p\mathbb{R}^n\,, \quad (d\chi_U)_p : T_p M \longrightarrow T_p\mathbb{R}^n$$

**Proposition 1.** *The following are equivalent*

(*i*) $D_\gamma = D_\alpha$,

(*ii*) $(\chi_U \circ \gamma)'(0) = (\chi_U \circ \alpha)'(0)$,

(*iii*) $(d\chi_U)_p D_\gamma = (d\chi_U)_p D_\alpha$.

*Proof.* Exercise. □

**Proposition 2.** $(d\chi_U)_p$ *is a linear isomorphism.*

*Proof.* To be presented next meeting. □

We will conclude this discussion with an important generalization of Proposition 2 describing how tangent spaces can interact via differentiable maps. To do this, we will expand our previous definition of *differential* to encompass that of all differentiable maps: given a map $\varphi : M \to N$ between manifolds, we define the **differential** of $\varphi$ at a $p \in M$ to be the following map between tangent spaces:

$$\begin{aligned} (d\varphi)_p : T_p M &\longrightarrow T_{\varphi(p)} N \\ D_\gamma(-) &\longmapsto D_{\varphi\circ\gamma} \end{aligned}$$

As an exercise for the reader, convince yourself that when $M = U$, $N = \mathbb{R}^n$ and $\varphi = \chi_U$, this recovers our previous definition for $(d\chi_U)_p$.

**Proposition 3.** $(d\varphi)_p$ *is a linear morphism.*

## 1.2 $TM$ Locally

We will pick up here after the break.

## 1.3 $TM$ Globally As A Vector Bundle

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# References

[1] Brocker, tom Dieck, *Representations of Compact Lie Groups*. Springer, 1985.